18.453 lecture 4/5

Lecture plan:

1. non-bipartite matchings.

2. Tutte-Berge

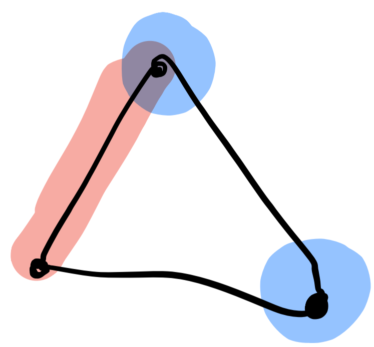
3. Algorithmic proof (Edmonds' alg)

* might not finish!

Non-bipartite Matching

- Given $G=(V,E)$;
  do <u>not</u> assume bipartite.
- want maximum matching M in G.
- König's theorem doesn't hold:

  max matching $\lneq$ min vertex cover.



- Recall from lecture 1: instead, duality w/ obstructions based on parity.

Tutte-Berge Formula

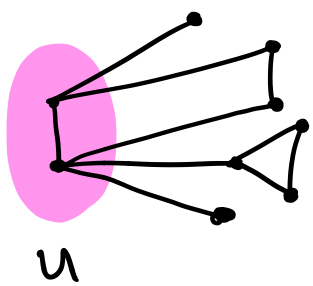
Given $U \subseteq V$,

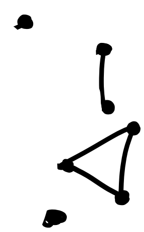
Def

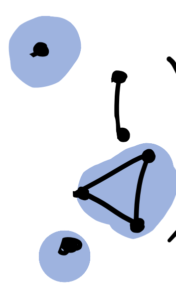
$G\setminus U :=$ G after deleting U & all adjacent edges.

$o(G\setminus U) :=$ # odd connected components in $G\setminus U$ (# c.c.'s w/ odd # of verts).

E.g.

$G =$ 

$G\setminus U =$ 

$o(G\setminus U) = o\Big($  $\Big) = 3$

Thm (Tutte-Berge Formula):

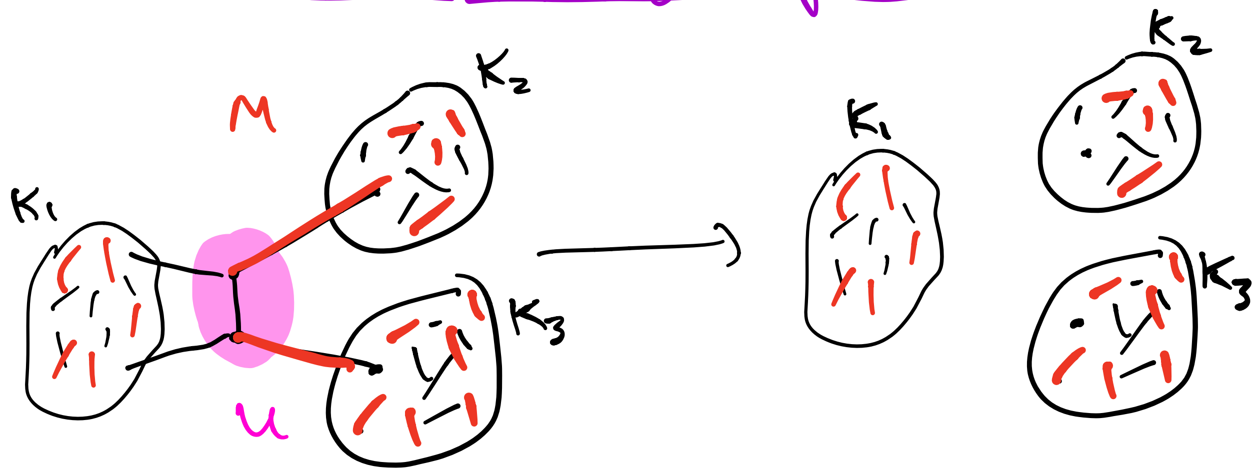
$$\max_{\text{matching } M} \underbrace{|M|}_{\#\text{ edges}} = \min_{U \subseteq V} \frac{1}{2}\Big(|V| + |U| - o(G \setminus U)\Big)$$

not there in lec 1

<u>Pf</u> $(\leq)$ i.e. "<u>weak duality</u>"

- Deleting U deletes $\leq |U|$ edges of M.

How many left over?



- here, # left over is at most

$$\sum_{i=1}^{3} \left\lfloor \frac{|K_i|}{2} \right\rfloor$$

- Thus, if $K_1 \dots K_k$ are connected components of $G \setminus U$,

$$* \quad |M| \le |U| + \sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor.$$

- Can rewrite:

$$\left\lfloor \frac{|K_i|}{2} \right\rfloor = \begin{cases} \frac{|K_i|}{2} & \text{if } |K_i| \text{ even} \\ \frac{|K_i|-1}{2} & \text{else.} \end{cases}$$

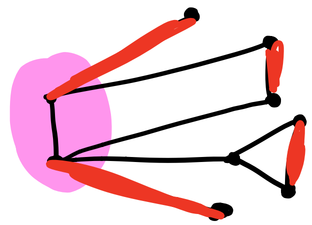
thus $** \quad \displaystyle\sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor = \underbrace{\sum_{i=1}^{k} \frac{|K_i|}{2}} - \frac{1}{2} o(G \setminus U).$

$$= \frac{|V| - |U| - o(G \setminus U)}{2}$$

- Plugging ** into * gives

$$|M| \le \frac{|V|}{2} + \frac{|U|}{2} - \frac{o(G \setminus U)}{2}. \qquad \square$$

E.g.

$G =$ 

U

$|M| = 4$, $\quad \frac{1}{2}\left(|V| + |U| - o(G \setminus U)\right)$
$= \frac{1}{2}(9 + 2 - 3) = 4.$

Proof of $\geq$?

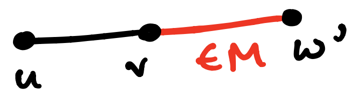
- Beautiful algorithm due to Edmonds.
- <u>Challenge</u>: though still true that

  M max $\iff$ no augmenting path w.r.t M.

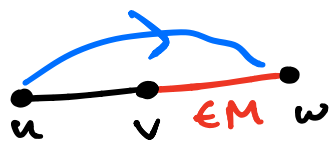
  <u>finding</u> the paths is hard.

- Why? Natural approach repeats vertices. (Fails)

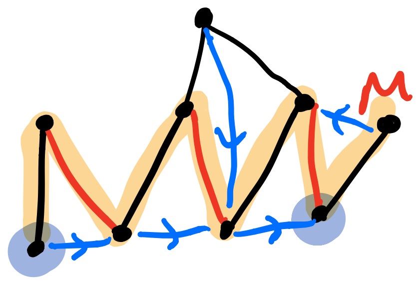
<u>Natural approach</u>: whenever you see



, add directed edge uw:



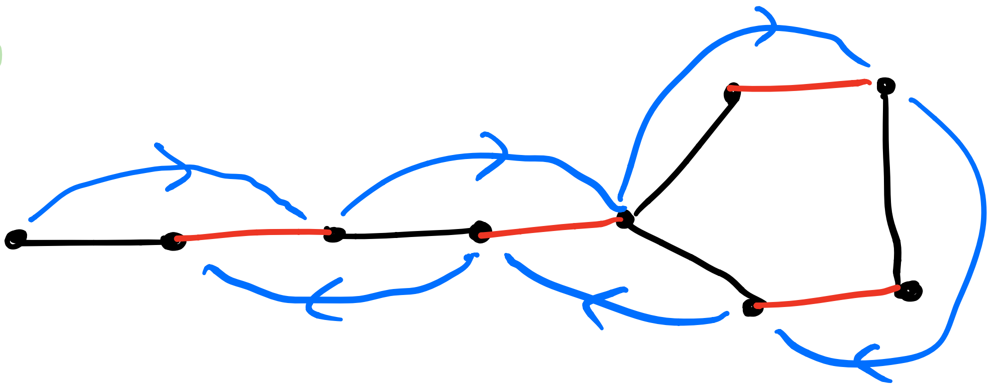
<u>E.g.</u>



Then, <u>start</u> at exposed vertex & look for vertex adjacent to an exposed vertex in blue digraph.

<u>Problem</u>: can lead to repeated vertices.

E.g.



when we first repeat, have found a

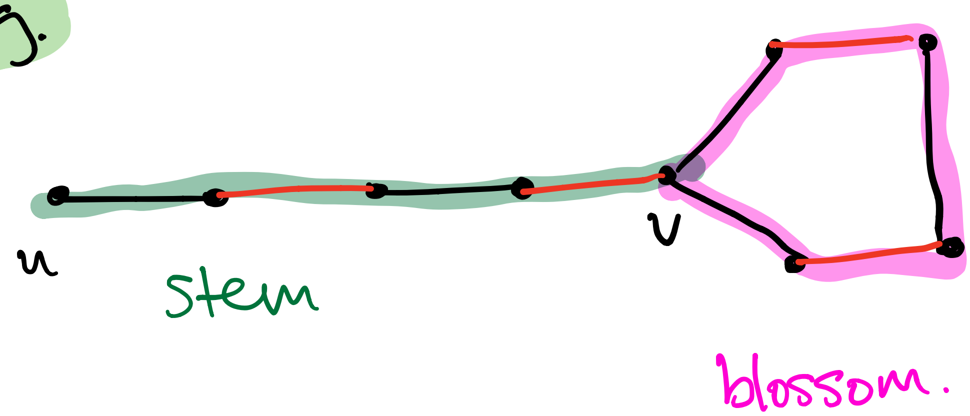
flower (with respect to M):

Stem: even-length alternating path from exposed vertex u to vertex v

+

Blossom: odd-length cycle intersects stem in v alternating *except* for edges incident to v.

E.g.



Algorithm idea:

At each step, have matching M.

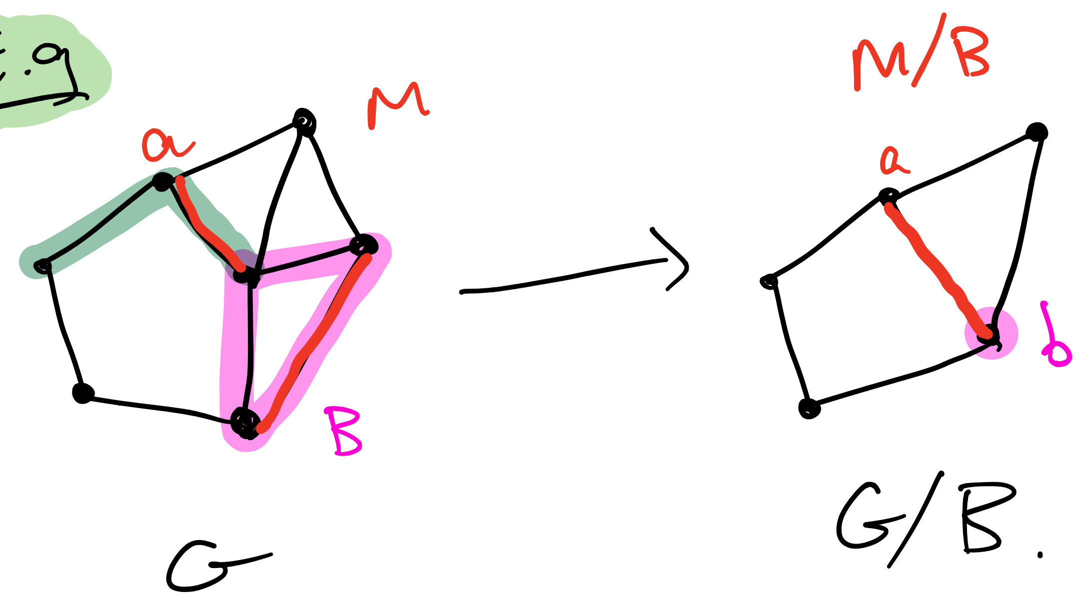
- find aug. path or flower w.r.t M or show neither exists.
- If neither exists, Matching is maximum. (b/c no aug. path)
- if aug. path, augment & repeat.

- if flower, let B be blossom.

Create graph G/B (not G\B) called contraction where

1. B shrunk to single vert. b
2. edges (u,v) $u \notin B$, $v \in B$ replaced by $(u,b) \in G/B$.

E.g



Note: is matching M/B in G/B

and $|M| - |M/B| = \frac{|B|-1}{2}$

(i.e. # edges of M in B).

Crucial Theorem: Let B be a blossom w.r.t. M. Then

$$\boxed{M \text{ max matching in } G} \iff \boxed{M/B \text{ max matching in } G/B.}$$

Proof will be **algorithmic**:

If bigger matching in G/B than

M/B, can use it to find bigger matching in G than M.

Theorem → Algorithm: recursion!

Assuming we can find either aug path or blossom, can recurse to increase size of M/B in G/B.

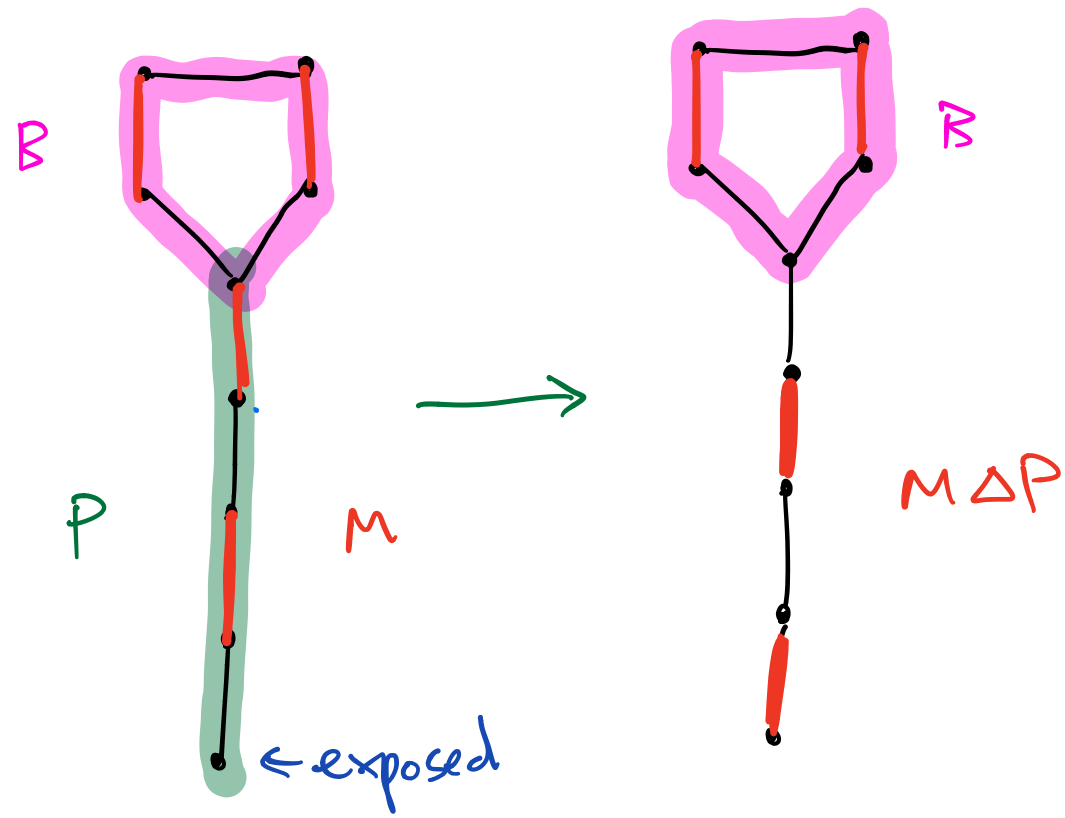
- if not possible: M maximum.
- else: use new matching in G/B to increase M

Proof of Crucial Theorem!

0. W.L.O.G. assumption:

B has empty stem!

why w.l.o.G? If P nonempty, look at M△P.



- M△P has empty stem & blossom B.

- Proving theorem for $M \Delta P$ also proves for M:

M maximum in G

P alternating $|M| = |M \Delta P|$

$\iff$ $M \Delta P$ maximum in G

Theorem for empty stem

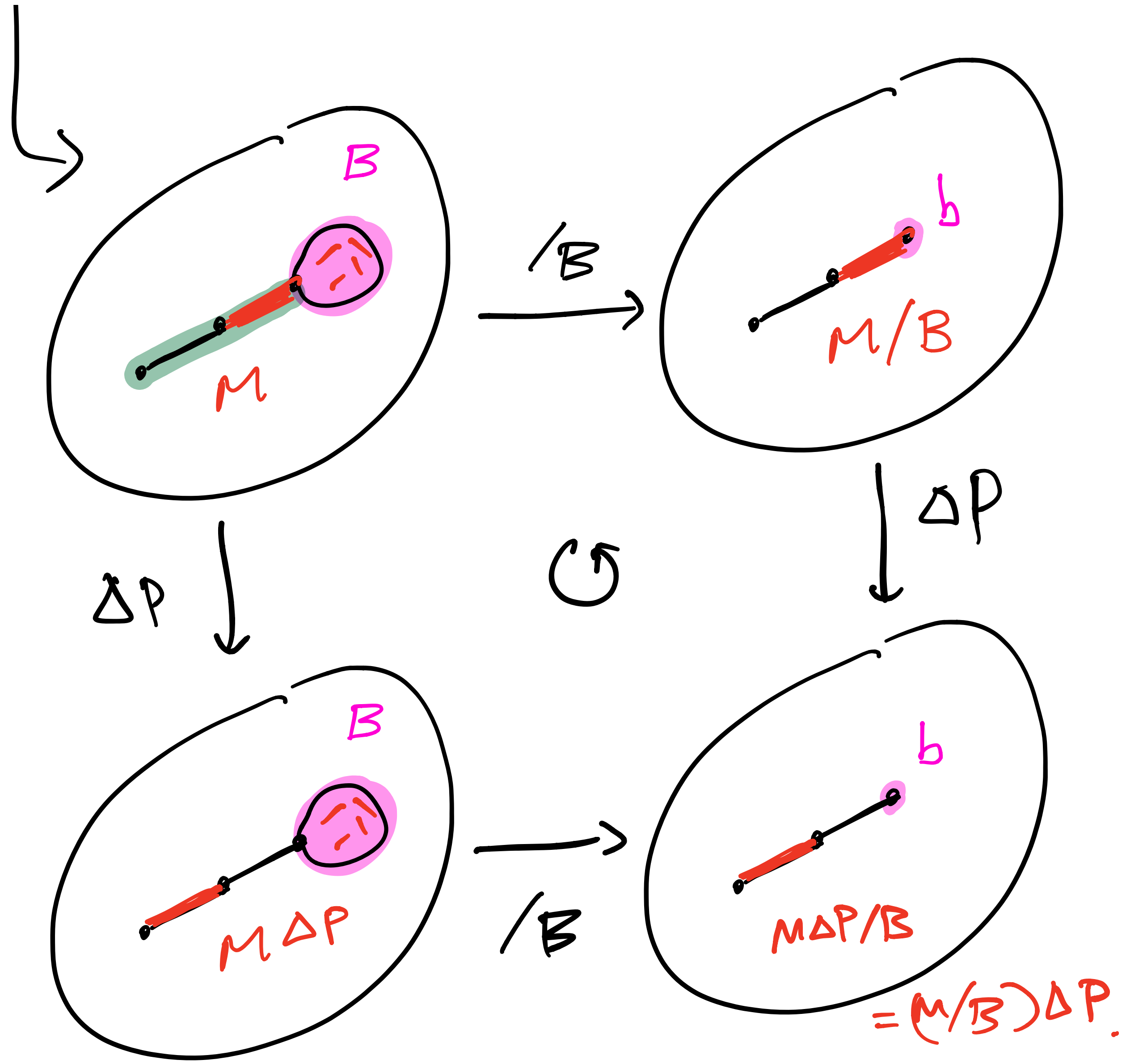
$\iff$ $M \Delta P / B$ max in G/B

$M \Delta P/B = (M/B) \Delta P$

$\iff$ $(M/B) \Delta P$ max in G/B

P still alternating.

$\iff$ (M/B) max in G/B.



Finally, start proof of <u>crucial thm</u>:

$\boxed{M \text{ max. in } G}$

$\iff$ $\boxed{M/B \text{ max in } G/B}$

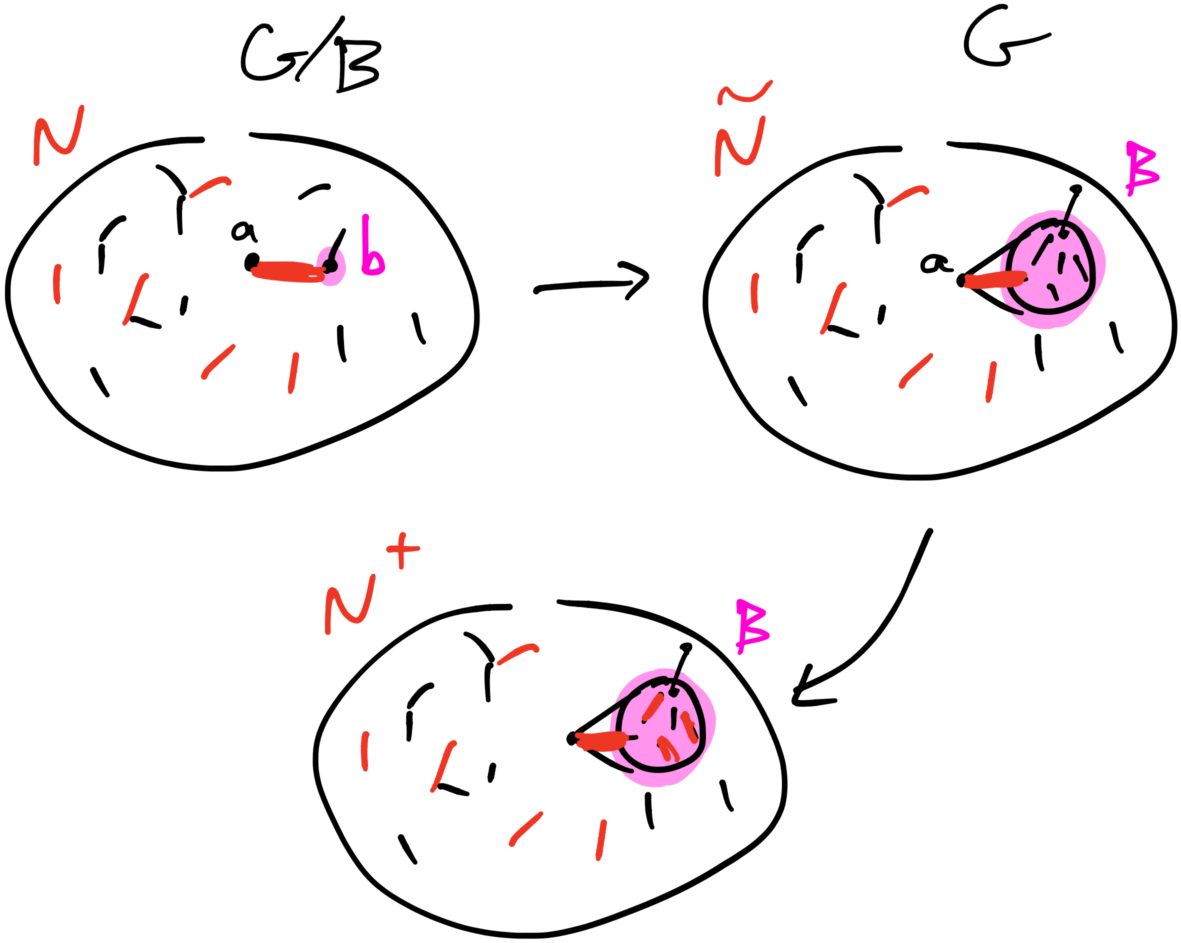
1. ($\Rightarrow$):

Contrapos: Suppose M/B not max, show M not max.

Suppose N is matching in G/B larger than M/B.

- pull back N to matching $\tilde{N}$ in G: $\tilde{N}/B = N$
  $\tilde{N}$ incident to ≤ 1 vertex of B.
- Expand to matching $N^{\uparrow}$ in G: add $\frac{1}{2}(|B|-1)$

edges in B.



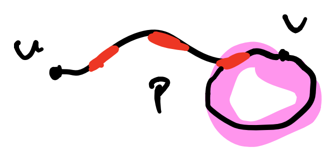
$|N^+|$ exceeds $|M|$ by same amt. $|N|$ exceeds $|M/B|$.

2. ($\Leftarrow$)

contrapos: if M not max, then M/B is not max.

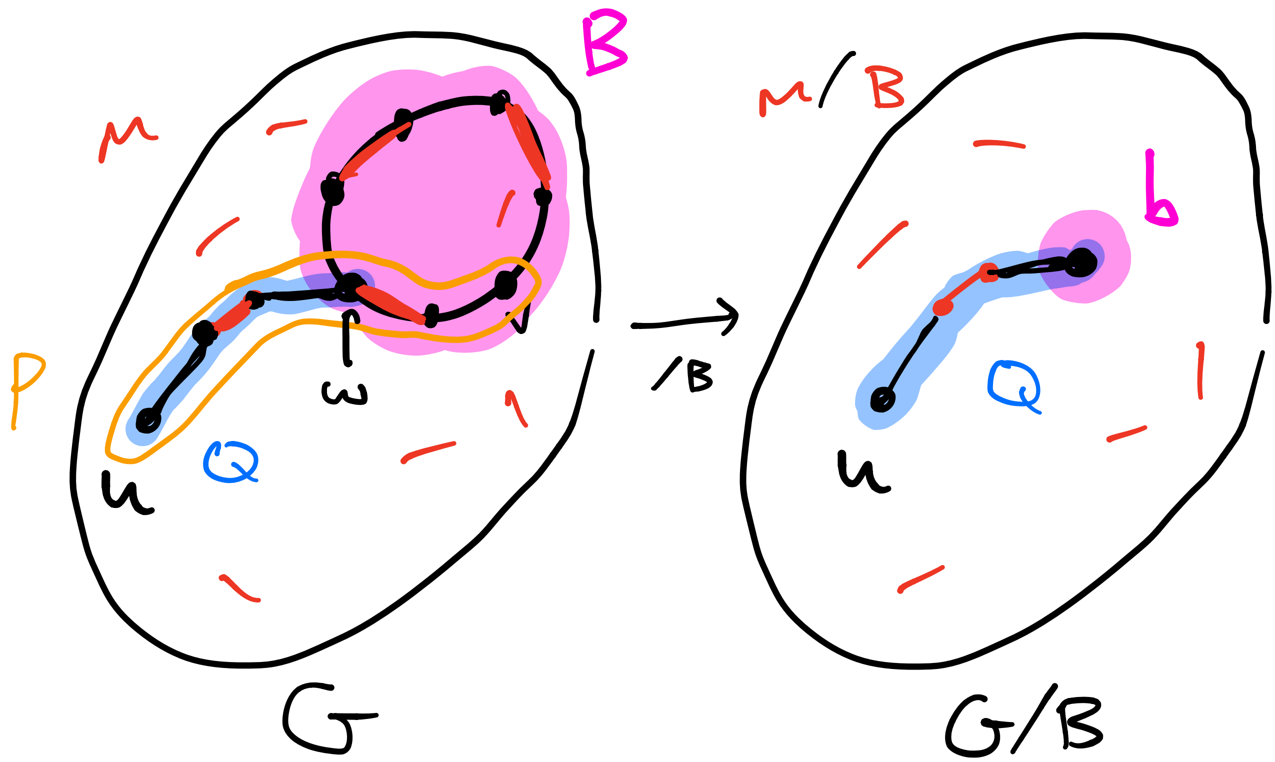
Suppose M not max in G.

- then $\exists$ aug path P between exposed verts $u, v \in G$.



- wlog $u \notin B$, B has only 1 exposed vertex. (stem is empty).
- $$w := \begin{cases} \text{first vertex of } P \text{ in } B \text{ (starting from } u\text{)} \\ v \quad \text{if } P, B \text{ share no vertices.} \end{cases}$$
- $Q :=$ part of P between u, w.
- Q augmenting path in M/B

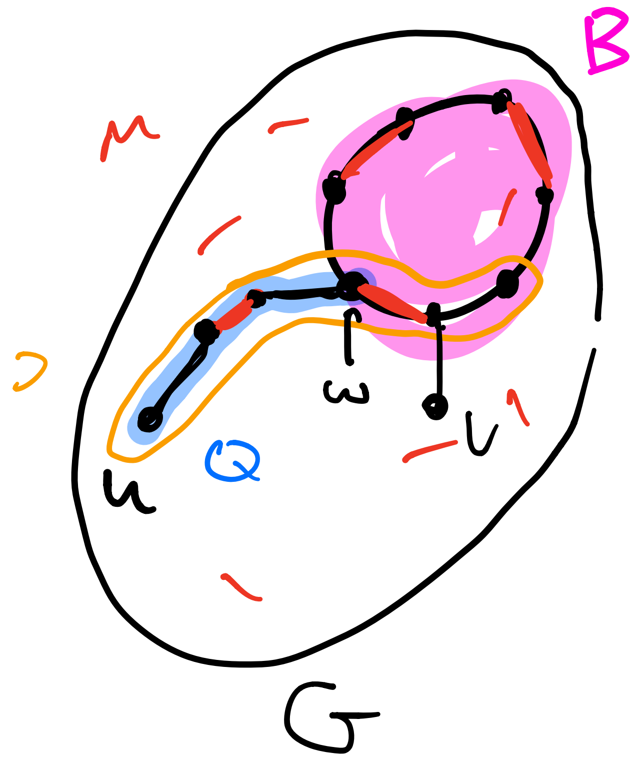
b exposed in M/B bc stem is empty.



Also: if P, B vertex disjoint, then P aug. M/B.

Finally: Augmenting M/B along Q

$\Rightarrow$ M/B not maximum $\square$



$\hat{M}/B = M^*$

$\uparrow$ B is a blossom for M

maybe not for $\hat{M}$.

<u>Subtlety</u>: thm <u>doesn't</u> say

maximum matching M^* in G/B

$\rightsquigarrow$ max matching $\hat{M}$ in G
by adding $\frac{|B|-1}{2}$ edges
from B to M^*!

Ex. find example of above:
i.e. blossom B of M so that M^* max
in G/B but adding the $\frac{|B|-1}{2}$ edges
to M^* in G doesn't result in max matching $\hat{M}$
in G

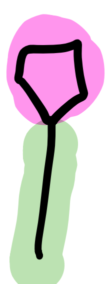
explain why no contradiction.

Lecture 5

Plan:

1. Finish Edmonds' alg.
2. Prove Tutte-Berge
3. (Maybe) Start polyhedra.

Announcements:



- I may be 10-15 mins late Thurs. (will keep you posted on slack).
- HW due 11:00pm Thurs.

Edmonds' Algorithm

(Given M, find augmenting path/flower)

- Label exposed vertices EVEN;

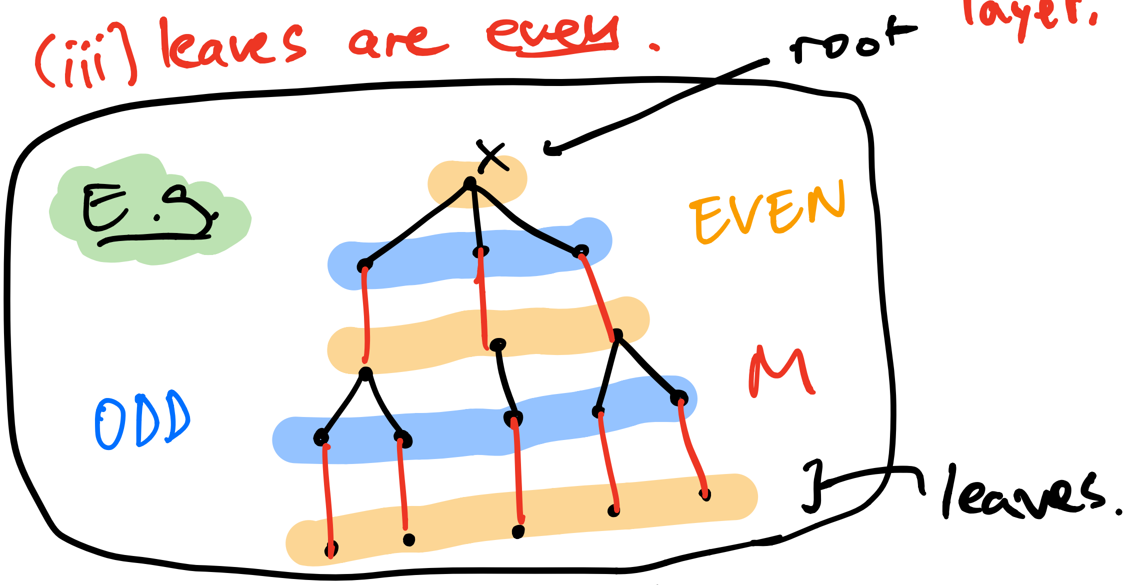
Keep others unlabelled initially.
(eventually will label some EVEN/ODD)

- Maintain alternating forest:

(sub) graph in which each connected component is alternating tree (AT)

i.e. tree s.t. paths to root are

(i) alternating w.r.t M
(ii) alternate b/w odd & even.
(iii) leaves are even.
(iv) only >1 child at even layer.



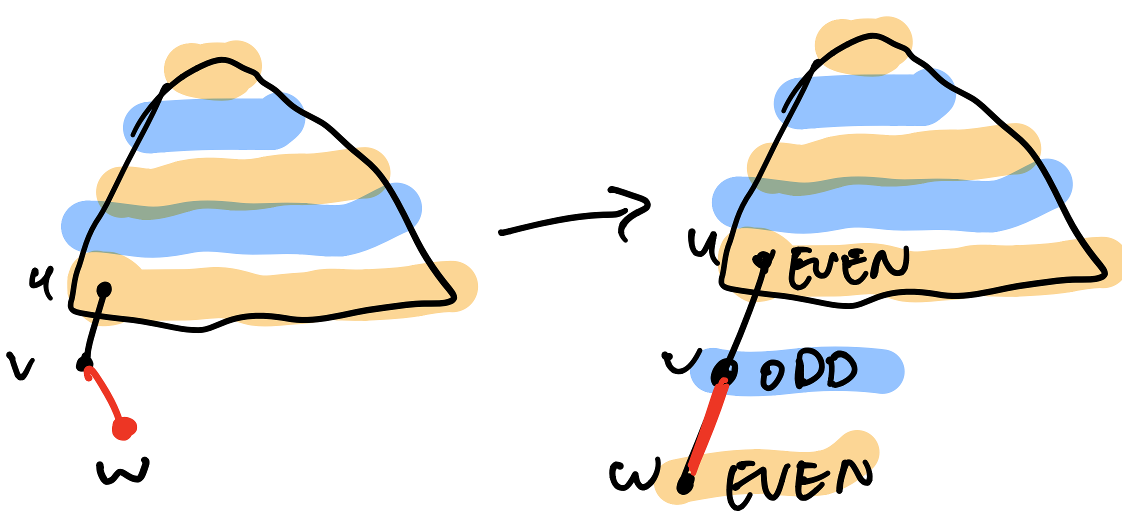
- Process EVEN vertices one at a time.

- currently on $u \in$ EVEN.
- all verts. in a tree are exposed or matched within the tree.

(a) If edge (u,v) with v unlabelled, label v ODD. $\boxed{v \text{ not exposed.}}$ (b/c else v EVEN);

label v's mate w EVEN.

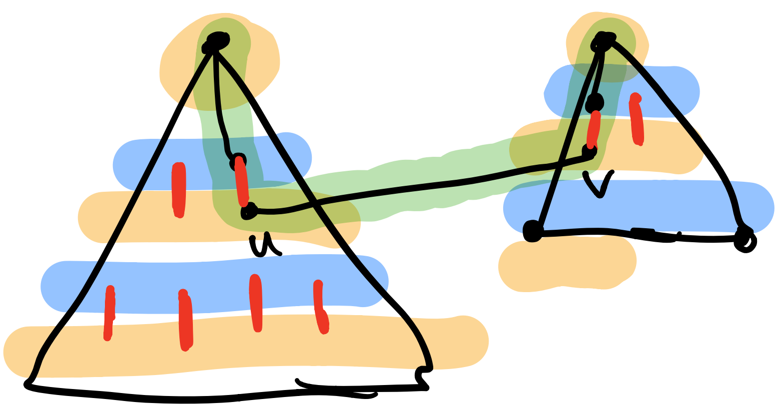
Add



(b) if $\exists$ edge (u,v) s.t.

v <u>EVEN</u> and v belongs to different AT than u,

Then is augmenting path between the roots.



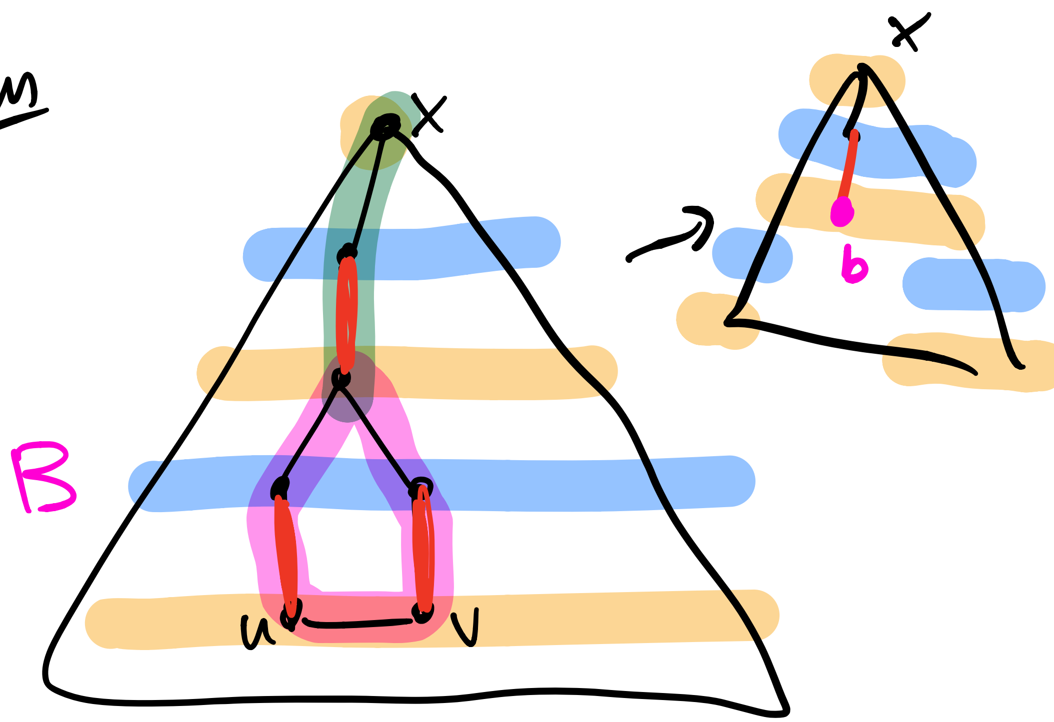
have found aug path; increase M, start over with new M.

(c) If is edge (u,v)
with v labeled **EVEN**
& v in u's AT,

---

then: the paths u, v → root
form a flower with (u,v).

flower
has <u>even</u>
stem



(c, cont.) Shrink to G/B.

- Keep same labelling & label b EVEN
- Recursively find max matching M^* in G/B.

Using crucial theorem, can use M^* to increase M.

Start over w/ new M.

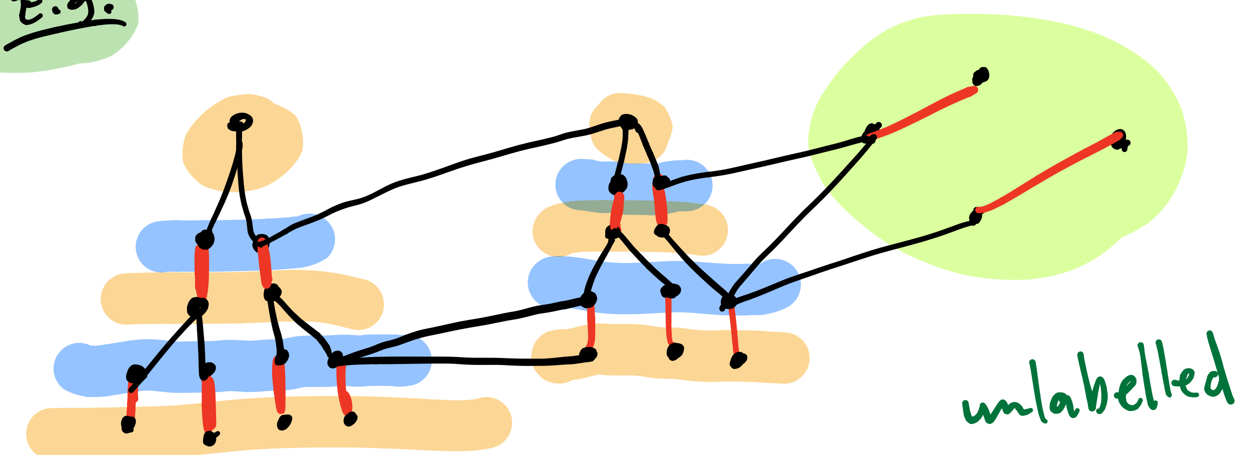
- if none (a, b, c) apply, then M is maximum.

Correctness: Suppose none of a, b, c apply anymore for the EVEN vertices. u.

Recall: a) (u,v) v unlabelled

b,c) (u,v) v, EVEN.

E.g.



note: (i) unlabelled matched to eachother.
(b/c AT verts. either exposed or matched w/in AT.

(ii) ODD vertices all matched to even vertices.

Claim: Current matching M_k is max in current $G_k = (V_k, E_k)$

$$G_k := \underbrace{G/B_1/B_2\ldots}_{G_i}/B_k$$

B_i blossom wrt M_{i-1} in G_{i-1}

Proof of Claim: Consider $U = \text{ODD}$ and consider the upper bound from Tutte-Berge for G_k,

$$|M_k| \underset{=}{\le} \frac{1}{2}\left[|V_k| + |U| - o(G_k/U)\right]$$

- No edges b/w EVEN vertices, (else (b) or (c) applies). & no edges b/w EVEN & unlabelled, (else (a) applies).

- Thus, EVEN are singleton components in $G_k \setminus U$, so $o(G_k \setminus ODD) \geq |EVEN|$ *

- All ODDS matched to EVENS & all unlabeled matched to unlabeled, so

$$** \begin{cases} |M_k| = |ODD| + \frac{1}{2}(|V_k| - |ODD| - |EVEN|) \\ = \frac{1}{2}(|V_k| + |ODD| - |EVEN|) \end{cases}$$

- Plug * into ** :

$$|M_k| = \frac{1}{2}(|V_k| + |ODD| - |EVEN|)$$

$$\geq \frac{1}{2}(|V_k| + |ODD| - o(G_k \setminus ODD)).$$

$$\frac{\text{Tutte Berge (upper bound)}}{M_k \text{ max in } G_k} \Rightarrow$$

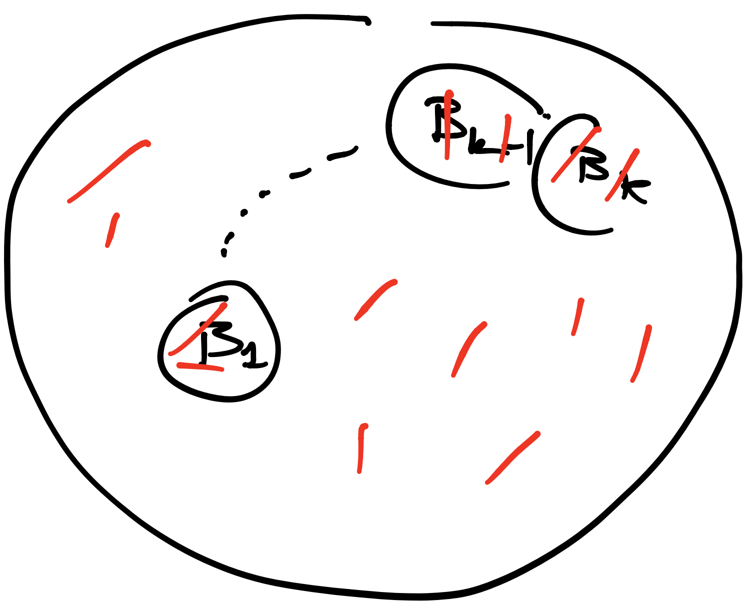
; claim proven.

Apply

crucial theorem repeatedly

for $B_k \, B_{k-1} \dots B_1$

<u>Shows</u>: algorithm constructs a max matching in G because B_i blossom for G_{i-1} M_{i-1}.



$\square$

Running time.

- Algorithm performs $\boxed{\leq \frac{n}{2}}$ augmentations of matching ("outer loop")
- between two augmentations ("inner loop") shrinks blossom $\boxed{\leq \frac{n}{2}}$ times (shrinking removes ≥ 2 vertices).
- Time to construct AT is $\boxed{O(m)}$, $m := |E|$.

So overall, $\boxed{O(n^2 m)}$.

Proof of Tutle-Berge $\geq$

We showed: TB holds for graph G_k for which alg. terminates.

want to find M s.t. $|M| \geq \frac{1}{2}(|V|+|U| - o(G/U))$.

- Recall

  G_i: $G/B_1 \dots /B_{i-1}$

  M_i: $M/B_1/\dots/B_{i-1}$

  G_0: G.

- TB holds for G_k, i.e.

$$|M_k| = \frac{1}{2}\left(|V_k| + |U| - o(G_k/U)\right)$$

where $U = \underline{ODD}$,

b/c $G_k \setminus \underline{ODD} = \underline{EVEN}$;

evens singleton CC.

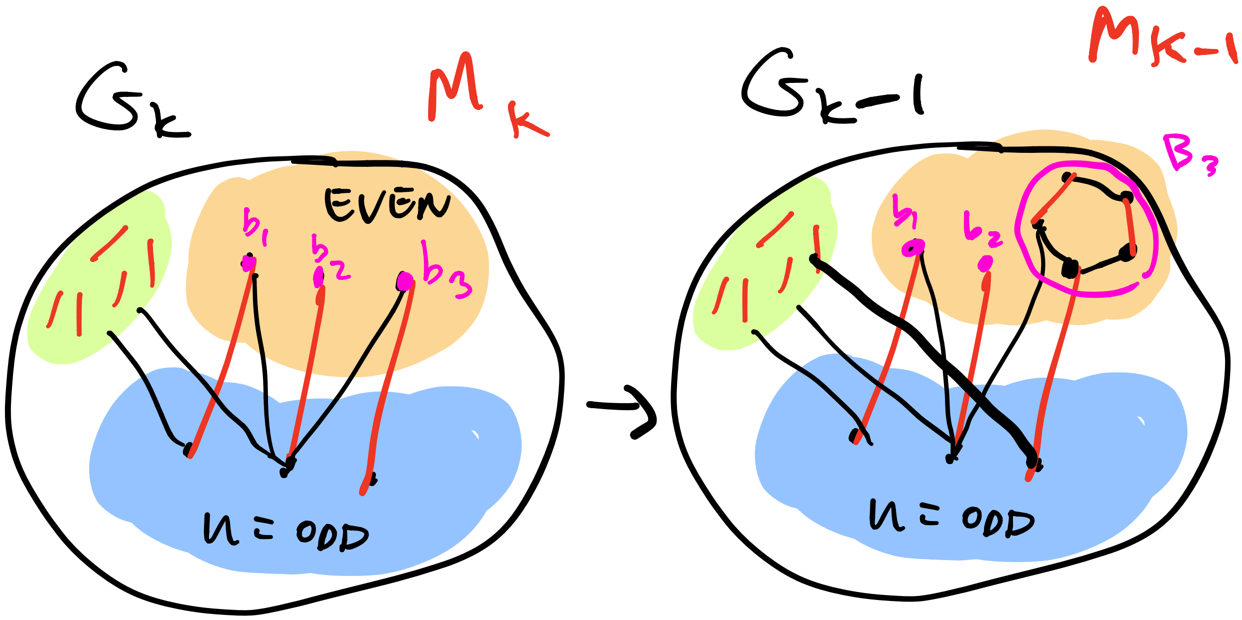
- Unshrink B_i one by one.

Claim: $U = \underline{ODD}$ obtains

equality in TB for all G_i, M_i:

$$|M_i| = \frac{1}{2}\left(|V_i| + |U| - o(G_i/U)\right).$$

backwards induction.



$$\boxed{\text{In step } G_i \to G_{i-1}:}$$

(i) $|V_{i-1}| = |V_i| + |B_i| - 1$

b_i itself

and

$$|M_{i-1}| = |M_i| + \frac{1}{2}(|B_i| - 1).$$

(ii) Unshrinking B_i adds even # $(|B_i| - 1)$ vertices to some C.C. of $G_i \setminus U$, so # odd/even components stays same.

i.e. $\boxed{o(G_{i-1} \setminus U) = o(G_i \setminus U)}$

(iii) Using this, when $i \leftarrow i-1$

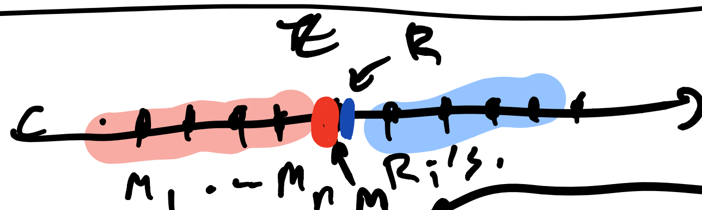
the RHS & LHS of

$$|M_i| = \frac{1}{2}\left(|V_i| + |U| - o(G_i \setminus U)\right)$$

increase by $\frac{1}{2}(|B_i| - 1)$.

Apply induction to conclude $\square$.

TB:



R_i

$$\max|M| = \min \frac{1}{2}\left(|V| + |U| - o(G \setminus U)\right).$$

(i) $|M| \le \frac{1}{2}\left(|V| + |U| - o(G \setminus U)\right)$

(ii) here: $\exists$ some M

$$|M| = \frac{1}{2}\left(|V| + |U| - o(G \setminus U)\right).$$

Corollary of Tutte-Berge:

G has p.m. iff

$\forall U, \ o(G \setminus U) \leq |U|$.

this is called

Tutte's matching thm.